

Last time...

Goal: Construct Θ -stratifications

$$\mathcal{X} = \mathcal{X}^{ss} \cup \bigcup_{\alpha} S_{\alpha}$$

along with good moduli space $q: \mathcal{X}^{ss} \rightarrow M$

Keywords from first lecture:

filtrations $F: \Theta \rightarrow \mathcal{X}$, numerical invariant $\mu(F)$
stability, HN problem

Thm A (HL):

if \mathcal{X} Θ -reductive, HN problem has a solution,
and only finitely many HN types in bounded family
 $\Rightarrow \mu$ defines Θ -stratification

Thm B (Alper-HL-Heinloth): if \mathcal{X} is bounded,
it has a GMS iff

- 1) \mathcal{X} is Θ -reductive, 2) unpunctured inertia,
- 3) closed points have reductive autom. groups.

Today: Discuss this theorem and applications to moduli of sheaves on a K3 surface

What is a Θ -reductive stack?

Def: A stack is Θ -reductive if...

For any family over DVR $\text{Spec}(R) \rightarrow \mathcal{X}$,
any filtration of the generic point extends
to a filtration of the family

Ex 1: Y projective scheme, $\mathcal{X} = \text{Coh}(Y)$
 \rightsquigarrow amounts to compactness of flag scheme
 \rightsquigarrow fails for $\text{Bun}(Y)$

Ex 2: more generally, proof can be adapted
to $\mathcal{X} = \{\text{objects in } \mathcal{A}\}$, \mathcal{A} abelian category

Ex 3: quotient stacks $\text{Spec}(A)/G$
 \uparrow reductive

Prop: if \mathcal{X} is Θ -reductive and a numerical
invariant μ defines a Θ -stratification, then \mathcal{X}^{ss}
is Θ -reductive.

Modifications and unpunctured inertia

Def: given a family over a DVR, we say that another map is a modification if the maps are isomorphic over the generic point $\text{Spec}(K)$

$$\text{Spec}(R) \rightarrow \mathcal{X}$$

Ex: family of bundles on $C \times \text{Spec}(R)$,
 $\mathcal{E}_0 =$ special fiber, $\mathcal{F} \subset \mathcal{E}_0$ sub-bundle
 $\mathcal{E}' = \ker(\mathcal{E} \rightarrow i_{*}(\mathcal{E}_0/\mathcal{F}))$

new bundle, elementary modification
 \leadsto can formulate a notion for arbitrary \mathcal{X}

Def: \mathcal{X} has unpunctured inertia if for any family over a DVR, one can find an elementary modification such that

- 1) any connected component of $\text{Aut}(\text{generic fiber})$ specializes to special fiber, or
- 2) an n -th finite-order generic automorph. specializes

Amplifications

If $q: \mathcal{X} \rightarrow M$ is a good moduli space, then

$\longrightarrow M$ is separated iff any modification over a DVR can be factored into sequence of elementary modifications

$\longrightarrow M$ is proper if it is separated and \mathcal{X} satisfies existence part of val. crit.

Thm (semistable reduction):

Given a Θ -stratification of \mathcal{X} , any family $\text{Spec}(R) \rightarrow \mathcal{X}$ with semistable generic point is related by elementary modification to a semistable family

Consequence: Can specify conditions on \mathcal{X} s.t. the good moduli space of \mathcal{X}^{ss} is proper

Slope semistability

Set up: $\mathcal{A} \subset D^b(Y)$ heart of bounded t-structure
 $\sigma \left\{ \begin{array}{l} \nu: K_0(Y) \rightarrow \Lambda = K_0^{\text{num}}(Y) \\ \zeta: \Lambda \rightarrow \mathbb{C}, \quad -\deg(\nu) + i \text{rk}(\nu) := \zeta(\nu) \end{array} \right.$

Pre-stability condition \Leftrightarrow all $E \in \mathcal{A}$ have HN filtrations

Hypothesis:

$\mathcal{X}_\nu(B) := \left\{ \begin{array}{l} E \in D^b(Y \times B) \text{ s.t.} \\ E|_b \in \mathcal{A} \text{ for all } b \in B \end{array} \right\}$
is an algebraic stack

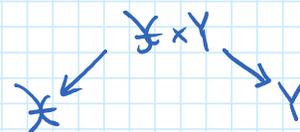
Ex 1: Slope semistability in $\text{Coh}(Y)$

Ex 2: Any Bridgeland stability condition with A noetherian

Rem: can work in more general categories, and can define \mathcal{X} directly from A

Moduli spaces

Central charge defines a line bundle: on \mathcal{X}_ν
write $z(E) = \chi(E \otimes \omega_z)$, $\omega_z \in K_0^{\text{num}}(Y) \otimes \mathbb{C}$



$$l \propto \text{ch}_1 \left(\bigoplus_{E \text{ univ}}^{Y \rightarrow \mathcal{X}} \left(\text{Im} \left(\frac{-\omega_z}{z(\nu)} \right) \right) \right)$$
$$b \propto \text{ch}_2 \left(\bigoplus_{E \text{ univ}}^{Y \rightarrow \mathcal{X}} \left(\text{Im}(\omega_z) \right) \right)$$

Consequences of main theorems: if $\mathcal{X}_\nu^{\text{ss}}$ bounded $\forall \nu \in \Lambda$, then

$\rightarrow \mathcal{X}$ has Θ -stratification by HN type

$\rightarrow \mathcal{X}_\nu^{\text{ss}}$ has proper good moduli space

Main example: From now on, we consider only Bridgeland stability on a smooth surface

\leadsto can study Donaldson invariants

Naive Donaldson invariants of surfaces

We will always work with K-theoretic invariants:

→ $\mathcal{F} \in K^0(\mathcal{X})$, e.g. $\mathcal{F} = \bigoplus_{\mathbb{Z}}^{\mathcal{Y} \rightarrow \mathcal{X}} (E)$
where $E \in K^0(\mathcal{Y})$, E obtained from $\mathcal{E}_{\text{univ}}$ by simple operations

→ $Rq_* : D^b(\mathcal{X}_v^{\text{ss}}) \rightarrow D^b(M_v^{\text{ss}})$ is well-defined by properties of GMS

Definition

$$I_v^\sigma(\mathcal{F}) := \chi(M_v^{\sigma\text{-ss}}, Rq_*(\mathcal{F})) = \chi(\mathcal{X}_v^{\sigma\text{-ss}}, \mathcal{F})$$

Question: how do $\mathcal{X}_v^{\sigma\text{-ss}}$ and $I_v^\sigma(E)$ depend on stability condition σ ? For nice results, we need to regard $\mathcal{X}_v^{\sigma\text{-ss}}$ as a derived stack.

→ Algebraic geometry built commutative DGA's
e.g. $A[\epsilon_1, \dots, \epsilon_r; d\epsilon_i = a_i \in A]$

→ $I_v^\sigma(\mathcal{F}) = \text{integral over derived stack}$

Correct Donaldson invariants of surfaces

A simple analogy for derived algebraic geometry:

reduced rings \rightsquigarrow rings \rightsquigarrow CDGA's

On affine objects:

$$H_0(A)_{\text{red}} \longleftarrow H_0(A) \longleftarrow A \quad \leftarrow \begin{array}{l} \text{assume} \\ \text{homologically} \\ \text{bounded} \end{array}$$

Analogous picture for derived schemes / stacks: $i : \mathcal{X}^{\text{cl}} \hookrightarrow \mathcal{X}$, same underlying points

Virtual structure sheaf: Note $\bigoplus H_i(A_0)$ is a coherent $H_0(A_0)$ -module
 \rightsquigarrow glue to define $\mathcal{O}_{\mathcal{X}}^{\text{vir}} := \bigoplus H_i(A_0)[i] \in D(\mathcal{X}^{\text{cl}})$

Classical shadow of derived world:

Given \mathcal{F} on derived stack \mathcal{X} ,
$$\begin{aligned} \chi(\mathcal{X}, \mathcal{F}) &= \chi(\mathcal{X}, \mathcal{F} \otimes_{L^*} \mathcal{O}_{\mathcal{X}}^{\text{vir}}) \\ &= \chi(\mathcal{X}^{\text{cl}}, L^*(\mathcal{F}) \otimes^L \mathcal{O}_{\mathcal{X}}^{\text{vir}}) \end{aligned}$$

Wall crossing

Situation: Fix $v \in K_0^{\text{num}}(S)$

→ σ varies in a complex manifold $\text{Stab}^*(S)$

→ in complement of real codim 1 walls,
 $\chi_v^{\sigma\text{-ss}}$ is constant

→ let $\sigma_0 \in \text{wall}$, σ_{\pm} in different chambers,
 $\chi_v^{\sigma_0\text{-ss}}$ has GMS, and $\chi_v^{\sigma_{\pm}\text{-ss}} \rightarrow \chi_v^{\sigma_0\text{-ss}}$
 in some cases, as in last lecture

$$\chi_v^{\sigma_0\text{-ss}} = \chi_v^{\sigma_{\pm}\text{-ss}} \cup \bigcup_{\alpha} \sum_{\alpha}^{\pm} \left\{ \begin{array}{l} v_1 + \dots + v_p = v \\ z(v_i) \in \mathbb{R}_{>0} z(v) \end{array} \right\}$$

Hypothesis: $\forall E \in A$, $L_{x,E}^{\text{vir}} = \text{RHom}(E, E[U])^*$
 has no cohomology in $\text{deg} < -1$, i.e.
 $\text{Hom}(E, E[i]) = 0$ for $i > 2$

Rem: Holds automatically when $A = \text{Coh}(S)$
 or when S is K3

Wall crossing formula

Thm: Under the previous hypotheses, we have

$$\begin{aligned} I_v^{\sigma_0}(\mathcal{F}) - I_v^{\sigma_{\pm}}(\mathcal{F}) \\ = \sum_{\alpha} \chi(\underbrace{\chi_{v_1}^{\sigma_1\text{-ss}} \times \dots \times \chi_{v_p}^{\sigma_1\text{-ss}}}_{\uparrow \text{centers of strata}}, \mathcal{F} \otimes E_{\alpha}^{\pm}) \end{aligned}$$

Decompose $L_x^{\text{vir}}|_{\text{center}} \cong L^+ \oplus L^0 \oplus L^-$
 $E_{\alpha} = \text{Sym}(L^-) \otimes \text{Sym}((L^+)^*) \otimes \det((L^+)^*)[-\text{rk}(L^+)]$

Proof idea:

Local cohomology, uses derived AG, and modular interpretation of the strata

Uses: 1) compute $I_v^{\sigma_+}(\mathcal{F}) - I_v^{\sigma_-}(\mathcal{F})$

2) come up with explicit formulas for $I_v^{\sigma_+}$ by wall crossing to where $\chi_v^{\text{ss}} = \emptyset$

Nagging question: Combinatorial structure?

Birational geometry -- K3 case

If v is primitive and σ generic, then

$$M_v^{\sigma-ss} = \text{smooth projective hyperkähler}$$

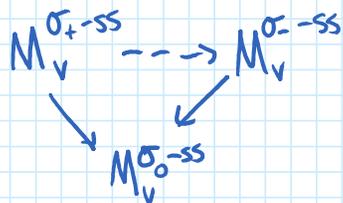
Restrict to class of CY manifolds birationally equivalent to $M_v^{\sigma-ss}$

Thm (Bayer-Macri): Any two manifolds in this class can be connected by a sequence of birational modifications of the form:

$$M_v^{\sigma_+-ss} \dashrightarrow M_v^{\sigma_--ss}$$

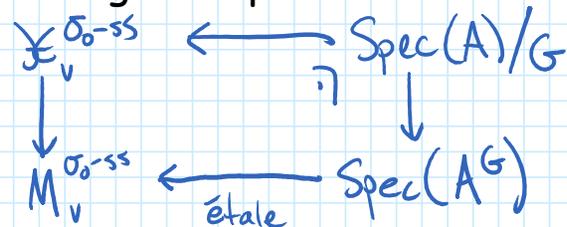
For some (twisted) K3 surface S

We now have a diagram:



Local models for flops

Base change the picture:



One can show, using self-duality $L_{\mathbb{A}} \cong L_{\mathbb{A}}^*$

Thm: $\text{Spec}(A)$ is zero fiber of a "weak" algebraic moment map $\mu: \text{Spec}(B) \rightarrow \mathfrak{g}^v$ for a smooth affine G -scheme.

Application: we recently used this to prove that any two smooth projective CY manifolds in birational class of $M_v^{\sigma-ss}$ have equivalent derived categories of coherent sheaves.